Mathematical Foundations:
Vectors, Matrices, & Parametric Equations

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Course web site: http://www.kddresearch.org/Courses/CIS636
Instructor home page: http://www.cis.ksu.edu/~bhsu

Readings:
McCausley (Senocular.com) tutorial: http://bit.ly/2yNPD

Lecture Outline

- Quick Review: Basic Precalculus and Linear Algebra for CG
- Matrix and Vector Notation, Operations
- Precalculus: Analytic Geometry and Trigonometry
  - Dot products and distance measures (norms, equations)
  - Review of some basic trigonometry concepts
- Vector Spaces and Affine Spaces
  - Subspaces
  - Linear systems, linear independence, bases, orthonormality
  - Equations for objects in affine spaces
- Cumulative Transformation Matrices (CTM) aka “Composite”, “Current”
  - Translation
  - Rotation
  - Scale
- Parametric Equation of Line Segment
Online Recorded Lectures for CIS 536/636 (Intro to CG)

- Project Topics for CIS 536/636
- Computer Graphics Basics (10)
  2. Graphics Pipeline – Week 2
  3. Detailed Introduction to Projections and 3-D Viewing – Week 3
  4. OpenGL Primer 1 of 3: Basic Primitives and 3-D – Weeks 3-4
  5. Rasterizing (Lines, Polygons, Circles, Ellipses) and Clipping – Week 4
  6. Lighting and Shading – Week 5
  7. OpenGL Primer 2 of 3: Boundaries (Meshes), Transformations – Weeks 5-6
  8. Texture Mapping – Week 6
  9. OpenGL Primer 3 of 3: Shading and Texturing, VBOs – Weeks 6-7
  10. Visible Surface Determination – Week 8
- Recommended Background Reading for CIS 636
- Shared Lectures with CIS 736 (Computer Graphics)
  - Regular in-class lectures (30) and labs (7)
  - Guidelines for paper reviews – Week 6
  - Preparing term project presentations, CG demos – Weeks 11-12

Background Expected

- Both Courses
  - Proficiency in C/C++ or strong proficiency in Java and ability to learn
  - Strongly recommended: matrix theory or linear algebra (e.g., Math 551)
  - At least 120 hours for semester (up to 150 depending on term project)
  - Angel’s OpenGL: A Primer recommended
- CIS 536 & 636 Introduction to Computer Graphics
  - Fresh background in precalculus: Algebra 1-2, Analytic Geometry
  - Linear algebra basics: matrices, linear bases, vector spaces
  - Watch background lectures
- CIS 736 Computer Graphics
  - Recommended: first course in graphics (background lectures as needed)
  - OpenGL experience helps
  - Read up on shaders and shading languages
  - Watch advanced topics lectures; see list before choosing project topic
Overview: First Month (Weeks 2-5 of Course)
- Review of mathematical foundations of CG: analytic geometry, linear algebra
- Line and polygon rendering
- Matrix transformations
- Graphical interfaces

Line and Polygon Rendering (Week 3)
- Basic line drawing and 2-D clipping
- Bresenham’s algorithm
- Follow-up: 3-D clipping, z-buffering (painter’s algorithm)

Matrix Transformations (Week 4)
- Application of linear transformations to rendering
- Basic operations: translation, rotation, scaling, shearing
- Follow-up: review of standard graphics libraries (e.g., OpenGL)

Graphical Interfaces
- Brief overview
- Survey of windowing environments (MFC, Java AWT)

Vector: Geometric Object with Length (Magnitude), Direction

Vector Notation (General Form)
- Row vector
- Column vector

Coordinates in \( \mathbb{R}^3 \) (Euclidean Space)
- Cartesian (see http://bit.ly/5z1UC) \( \mathbf{a} = (a_x, a_y, a_z) \).
- Cylindrical (see http://bit.ly/5v3u) \( \mathbf{v} = (r, \varphi, h) \).
- Spherical (see http://bit.ly/4CvMZ) \( \mathbf{v} = (\rho, \varphi, \iota) \).

Matrix: Rectangular Array of Numbers

\[
\begin{bmatrix}
0 & -1 & -2 & -3 \\
1 & 1 & -1 & -2 \\
2 & 1 & 0 & -1
\end{bmatrix}
\]
Determinants

- What Are Determinants?
  - Scalars associated with any square \((k \times k)\) matrix \(M, k \geq 1\)
  - Fundamental meaning: scale coefficient where \(M\) is linear transformation

- Definitions
  - \(2 \times 2\) matrix
    \[
    A = \begin{bmatrix}
    a & b \\
    c & d
    \end{bmatrix}
    \]
  - \(2 \times 2\) determinant
    \[
    \det A = ad - bc.
    \]
  - \(3 \times 3\) matrix
    \[
    A = \begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i
    \end{bmatrix}
    \]
  - \(3 \times 3\) determinant
    \[
    \det A = aei + bfg + cdh - ceg - afh - bdg.
    \]
    \[
    = a(ei - fh) - b(di - fg) + c(dh - eg)
    \]

General case (recursive definition): see http://mathworld.wolfram.com/Determinant.html

Vector Operations: Dot & Cross Product, Arithmetic

- Dot Product aka Inner Product aka Scalar Product
  \[
  \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
  \]

- Cross Product
  \[
  \mathbf{a} \times \mathbf{b} = \begin{vmatrix}
  i & j & k \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3
  \end{vmatrix}
  \]

- Vector Addition
  \[
  \mathbf{u} + \mathbf{v}
  \]
Matrix Operations [1]:
Scalar Multiplication & Transpose

- Scalar-Matrix Multiplication

\[
2 \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot (-3) \\ 2 \cdot 4 & 2 \cdot (-2) & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}
\]

- Transpose

\[
\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}
\]

Matrix Operations [2]:
Addition & Multiplication

- Matrix Addition

\[
\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 - 0 & 3 + 0 & 1 - 5 \\ 1 - 7 & 0 + 5 & 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ -6 & 5 & 0 \end{bmatrix}
\]

- Matrix Multiplication

\[
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\]

\[
B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}
\]

\[
AB = \begin{bmatrix} \sum a_{1j}b_{1j} & \sum a_{1j}b_{2j} & \cdots & \sum a_{1j}b_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum a_{mj}b_{1j} & \sum a_{mj}b_{2j} & \cdots & \sum a_{mj}b_{nj} \end{bmatrix}
\]

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Linear Systems of Equations

- **Definition:** Linear System of Equations (LSE)
  - Each of form \( a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \)
  - System shares same set of variables \( x_i \)

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots & \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\]

- **Example**
  - 3 equations in 3 unknowns
  - Solution:

\[
\begin{align*}
x &= 1 \\
y &= -2 \\
z &= -2
\end{align*}
\]

Cumulative Transformation Matrices: Basic T, R, S

- **T:** Translation (see [http://en.wikipedia.org/wiki/Translation_matrix](http://en.wikipedia.org/wiki/Translation_matrix))
  - Given
    - Point to be moved – e.g., vertex of polygon or polyhedron
    - Displacement vector (also represented as point)
  - Return: new, displaced (translated) point of rigid body

- **R:** Rotation (see [http://en.wikipedia.org/wiki/Rotation_matrix](http://en.wikipedia.org/wiki/Rotation_matrix))
  - Given
    - Point to be rotated about axis
    - Axis of rotation
    - Degrees to be rotated
  - Return: new, displaced (rotated) point of rigid body

- **S:** Scaling (see [http://en.wikipedia.org/wiki/Scaling_matrix](http://en.wikipedia.org/wiki/Scaling_matrix))
  - Given
    - Set of points centered at origin
    - Scaling factor
  - Return: new, displaced (scaled) point

Translation

- Rigid Body Transformation
- To Move \( p \) Distance and Magnitude of Vector \( v \):

\[
T_v p = \begin{bmatrix}
1 & 0 & 0 & v_x \\
0 & 1 & 0 & v_y \\
0 & 0 & 1 & v_z \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
p_x \\
p_y \\
p_z \\
1
\end{bmatrix} = \begin{bmatrix}
p_x + v_x \\
p_y + v_y \\
p_z + v_z \\
1
\end{bmatrix} = p + v.
\]

- Invertibility

\[
T_v^{-1} = T_{-v}.
\]

- Compositionality

\[
T_u T_v = T_{u+v}.
\]

Rotation

- Rigid Body Transformation
- Properties: Inverse = Transpose

\[
Q^T Q = I = Q Q^T \\
\text{det} Q = +1
\]

- Idea: Define New (Relative) Coordinate System
- Example

\[
Q = \begin{bmatrix}
0.8 & -0.6 & 0 \\
0.6 & 0.8 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- Rotations about \( x \), \( y \), and \( z \) Axes (using Plain 3-D Coordinates)

\[
Q_x(\theta) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}, \quad
Q_y(\theta) = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}, \quad
Q_z(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

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**Scaling**

- **Not Rigid Body Transformation**
- **Idea:** Move Points Toward/Away from Origin

$$S_v p = \begin{bmatrix} v_x & 0 & 0 & 0 \\ 0 & v_y & 0 & 0 \\ 0 & 0 & v_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} v_x p_x \\ v_y p_y \\ v_z p_z \\ 1 \end{bmatrix}$$

- **Homogeneous Coordinates Make It Easier**

$$S_v p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

- **Result**

$$\begin{bmatrix} sp_x \\ sp_y \\ sp_z \\ 1 \end{bmatrix}$$

- **Ratio Need Not Be Uniform in x, y, z**

![Scaling](https://commons.wikimedia.org/wiki/File:Scaling.png)

© 1993 Neider, Davis, Woo

[http://fly.cc.fer.hr/~unreal/theredbook/](http://fly.cc.fer.hr/~unreal/theredbook/)

**Other Transformations**

- **Shear:** Used with Oblique Projections
- **Perspective to Parallel View Volume (“D” in Foley et al.)**
- **See also**

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[http://www.bobpowell.net/transformations.htm](http://www.bobpowell.net/transformations.htm)
Quick Review: Basic Linear Algebra for CG

- A.1 Vector Spaces and Affine Spaces
  - Equations of lines, planes
  - Vector subspaces and affine subspaces
- A.2 Standard Constructions in Vector Spaces
  - Linear independence and spans
  - Coordinate systems and bases
- A.3 Dot Products and Distances
  - Dot product in \( \mathbb{R}^n \)
  - Norms in \( \mathbb{R}^n \)
- A.4 Matrices
  - Binary matrix operations: basic arithmetic
  - Unary matrix operations: transpose and inverse
- Application: Transformations and Change of Coordinate Systems

Vector Spaces and Affine Spaces

- **Vector Space**: Set of Points with Addition, Multiplication by Constant
  - Components
    - Set \( V \) (of vectors \( u, v, w \)) over which addition, scalar multiplication defined
    - Vector addition: \( v + w \)
    - Scalar multiplication: \( \alpha v \)
  - Properties (necessary and sufficient conditions)
    - Addition: associative, commutative, identity (0 vector such that \( \forall v . 0 + v = v \), admits inverses (\( \forall v . \exists w . v + w = 0 \))
    - Scalar multiplication: satisfies \( \forall \alpha, \beta, v . (\alpha \beta)v = \alpha(\beta v), \forall v . 1v = v, \forall \alpha, \beta, v . (\alpha + \beta)v = \alpha v + \beta v, \forall \alpha, \beta, v . \alpha(v + w) = \alpha v + \alpha w \)
    - Linear combination: \( \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \)
- **Affine Space**: Set of Points with Geometric Operations (No “Origin”)
  - Components
    - Set \( V \) (of points \( P, Q, R \)) and associated vector space
    - Operators: vector difference, point-vector addition
  - Affine combination (of \( P \) and \( Q \) by \( t \in \mathbb{R} \)): \( P + t(Q - P) \)
  - NB: for any vector space \( (V, +, \cdot) \) there exists affine space (points \( V \), \( V \))
Linear and Planar Equations in Affine Spaces

- **Equation of Line in Affine Space**
  - Let \( P, Q \) be points in affine space
  - **Parametric form** (real-valued parameter \( t \))
    - Set of points of form \((1 – t)P + tQ\)
    - Forms line passing through \( P \) and \( Q \)
  - **Example**
    - Cartesian plane of points \((x, y)\) is an affine space
    - Parametric line between \((a, b)\) and \((c, d)\):
      \[ L = \{(1 – t)a + tc, (1 – t)b + td \mid t \in \mathbb{R} \} \]

- **Equation of Plane in Affine Space**
  - Let \( P, Q, R \) be points in affine space
  - **Parametric form** (real-valued parameters \( s, t \))
    - Set of points of form \((1 – s)((1 – t)P + tQ) + sR\)
    - Forms plane containing \( P, Q, R \)

Vector Space Spans and Affine Spans

- **Vector Space Span**
  - Definition – set of all linear combinations of a set of vectors
  - **Example**: vectors in \( \mathbb{R}^3 \)
    - Span of single (nonzero) vector \( v \): line through the origin containing \( v \)
    - Span of pair of (nonzero, noncollinear) vectors: plane through the origin containing both
    - Span of 3 of vectors in **general position**: all of \( \mathbb{R}^3 \)

- **Affine Span**
  - Definition – set of all affine combinations of a set of points \( P_1, P_2, \ldots, P_n \) in an affine space
  - **Example**: vectors, points in \( \mathbb{R}^3 \)
    - Standard affine plan of points \((x, y, 1)^T\)
    - Consider points \( P, Q \)
    - Affine span: line containing \( P, Q \)
    - Also intersection of span, affine space
Independence

- **Linear Independence**
  - **Definition**: (linearly) dependent vectors
    - Set of vectors \( \{v_1, v_2, \ldots, v_n\} \) such that one lies in the span of the rest
    - \( \exists v_i \in \{v_1, v_2, \ldots, v_n\} \cdot v_i \in \text{Span} (\{v_1, v_2, \ldots, v_n\} \setminus \{v_i\}) \)
  - (Linearly) independent: \( \{v_1, v_2, \ldots, v_n\} \) not dependent

- **Affine Independence**
  - **Definition**: (affinely) dependent points
    - Set of points \( \{v_1, v_2, \ldots, v_n\} \) such that one lies in the (affine) span of the rest
    - \( \exists P_i \in \{P_1, P_2, \ldots, P_n\} \cdot P_i \in \text{Span} (\{P_1, P_2, \ldots, P_n\} \setminus \{P_i\}) \)
  - (Affinely) independent: \( \{P_1, P_2, \ldots, P_n\} \) not dependent

- **Consequences of Linear Independence**
  - Equivalent condition: \( \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \iff \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \)
  - Dimension of span is equal to the number of vectors

Subspaces

- **Intuitive Idea**
  - \( \mathbb{R}^n \): vector or affine space of “equal or lower dimension”
  - Closed under constructive operator for space

- **Linear Subspace**
  - **Definition**
    - Subset \( S \) of vector space \( (V, +, \cdot) \)
    - Closed under addition (+) and scalar multiplication (\( \cdot \))
  - **Examples**
    - Subspaces of \( \mathbb{R}^3 \): origin (0, 0, 0), line through the origin, plane containing origin, \( \mathbb{R}^3 \) itself
    - For vector \( v \), \( \\{\alpha v \mid \alpha \in \mathbb{R}\} \) is a subspace (why?)

- **Affine Subspace**
  - **Definition**
    - Nonempty subset \( S \) of vector space \( (V, +, \cdot) \)
    - Closure \( S' \) of \( S \) under point subtraction is a linear subspace of \( V \)
  - **Important affine subspace of \( \mathbb{R}^3 \):** \( \{(x, y, z, 1)\} \)
  - Foundation of homogeneous coordinates, 3-D transformations
Bases

- **Spanning Set** (of Set S of Vectors)
  - Definition: set of vectors for which any vector in Span(S) can be expressed as linear combination of vectors in spanning set
  - Intuitive idea: spanning set “covers” Span(S)

- **Basis** (of Set S of Vectors)
  - Definition
    - Minimal spanning set of S
    - **Minimal:** any smaller set of vectors has smaller span
  - Alternative definition: linearly independent spanning set

- **Exercise**
  - Claim: basis of subspace of vector space is always linearly independent
  - Proof: by contradiction (suppose basis is dependent... not minimal)

- **Standard Basis for \( \mathbb{R}^3 \)**:
  - \( E = \{ e_1, e_2, e_3 \} \), \( e_1 = (1, 0, 0)^T \), \( e_2 = (0, 1, 0)^T \), \( e_3 = (0, 0, 1)^T \)

- How to use this as coordinate system?

Coordinates and Coordinate Systems

- **Coordinates Using Bases**
  - Coordinates
    - Consider basis \( B = \{ v_1, v_2, \ldots, v_n \} \) for vector space
    - Any vector \( v \) in the vector space can be expressed as linear combination of vectors in \( B \)
    - **Definition:** coefficients of linear combination are coordinates
  - Example
    - \( E = \{ e_1, e_2, e_3 \} \), \( i = e_1 = (1, 0, 0)^T \), \( j = e_2 = (0, 1, 0)^T \), \( k = e_3 = (0, 0, 1)^T \)
    - Coordinates of \( (a, b, c) \) with respect to \( E: (a, b, c)^T \)

- **Coordinate System**
  - **Definition:** set of independent points in affine space
  - Affine span of coordinate system is entire affine space

- **Exercise**
  - Derive basis for associated vector space of arbitrary coordinate system
  - (Hint: consider definition of affine span...)
Dot Products and Distances

- **Dot Product in** \( \mathbb{R}^n \)
  - **Definition**
    - Dot product \( u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n \)
    - Also known as inner product
    - In \( \mathbb{R}^n \), called scalar product

- **Applications of the Dot Product**
  - Normalization of vectors
  - Distances
  - Generating equations
  - See Appendix A.3, Foley et al. (FVFH aka FVD)

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Norms and Distance Formulas

- **Length**
  - **Definition**
    - \( \|v\| = \sqrt{v \cdot v} \)
    - \( v \cdot v = \sum_i v_i^2 \)
  - aka Euclidean norm

- **Applications of the Dot Product**
  - Normalization of vectors: division by scalar length \( \|v\| \) converts to unit vector
  - Distances
    - Between points: \( \|Q - P\| \)
    - From points to planes
  - Generating equations (e.g., point loci): circles, hollow cylinders, etc.
  - Ray / object intersection equations
  - See A.3.5, FVD
### Orthonormal Bases

#### Orthogonality
- **Given:** vectors \( u = (u_1, u_2, \ldots, u_n)^T \), \( v = (v_1, v_2, \ldots, v_n)^T \)
- **Definition**
  - \( u, v \) are **orthogonal** if \( u \cdot v = 0 \)
  - In \( \mathbb{R}^2 \), angle between orthogonal vectors is 90°

#### Orthonormal Bases
- **Necessary and sufficient conditions**
  - \( B = \{b_1, b_2, \ldots, b_n\} \) is basis for given vector space
  - Every pair \( (b_i, b_j) \) is orthogonal
  - Every vector \( b_i \) is of unit magnitude (\( ||v_i|| = 1 \))

- Convenient property: can just take dot product \( v \cdot b_i \) to find coefficients in linear combination (coordinates with respect to \( B \)) for vector \( v \)

### Parametric Equation of a Line Segment

#### Parametric form for line segment
- \( X = x_0 + t(x_1 - x_0) \quad 0 \leq t \leq 1 \)
- \( Y = y_0 + t(y_1 - y_0) \)
- \( P(t) = P_0 + t(P_1 - P_0) \)

- “true,” i.e., interior intersection, if \( sedge \) and \( tline \) in \([0,1]\)
3 x 3 rotation matrices

We learned about 3 x 3 matrices that “rotate” the world (we’re leaving out the homogeneous coordinate for simplicity).

When they do, the three unit vectors that used to point along the x, y, and z axes are moved to new positions.

Because it is a rigid-body rotation
- the new vectors are still unit vectors
- the new vectors are still perpendicular to each other
- the new vectors still satisfy the “right hand rule”

Any matrix transformation that has these three properties is a rotation about some axis by some amount!

Let’s call three x-axis, y-axis, and z-axis-aligned unit vectors \( e_1, e_2, e_3 \)

Writing out:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

---

**Textbook and Recommended Books**

**Required Textbook**


**Recommended References**


Summary

- Cumulative Transformation Matrices (CTM): T, R, S
  - Translation
  - Rotation
  - Scaling
  - Setup for Shear, Perspective to Parallel – see Eberly, Foley et al.

- “Matrix Stack” in OpenGL: Premultiplication of Matrices

- Coming Up
  - Parametric equations in clipping
  - Intersection testing: ray-cube, ray-sphere, implicit equations (ray tracing)

- Homogeneous Coordinates: What Is That 4th Coordinate?
  - Crucial for ease of normalizing T, R, S transformations in graphics
  - See: Slide 22 of this lecture
  - Note: Slides 13 & 15 (T, S) versus 14 (R)
  - Read about them in Eberly 2e, Angel 3e
  - Special case: barycentric coordinates

Terminology

- Cumulative Transformation Matrices (CTM): Translation, Rotation, Scaling

- Some Basic Analytic Geometry and Linear Algebra for CG
  - Vector space (VS) – set of vectors admitting addition, scalar multiplication and observing VS axioms
  - Affine space (AS) – set of points with associated vector space admitting vector difference, point-vector addition and observing AS axioms
  - Linear subspace – nonempty subset S of VS (V, +, ·) closed under + and ·
  - Affine subspace – nonempty subset S of VS (V, +, ·) such that closure S’ of S under point subtraction is a linear subspace of V
  - Span – set of all linear combinations of set of vectors
  - Linear independence – property of set of vectors that none lies in span of others
  - Basis – minimal spanning set of set of vectors
  - Dot product – scalar-valued inner product <u, v> = u • v = u_1v_1 + u_2v_2 + … + u_nv_n
  - Orthogonality – property of vectors u, v that u • v = 0
  - Orthonormality – basis containing pairwise-orthogonal unit vectors
  - Length (Euclidean norm) – |v| = √v • v