Mathematical Foundations:
Vectors, Matrices, & Parametric Equations

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Course web site: http://www.kddresearch.org/Courses/CIS636
Instructor home page: http://www.cis.ksu.edu/~bhsu

Readings:
Lecture Outline

- Quick Review: Basic Precalculus and Linear Algebra for CG
- Matrix and Vector Notation, Operations
- Precalculus: Analytic Geometry and Trigonometry
  - Dot products and distance measures (norms, equations)
  - Review of some basic trigonometry concepts
- Vector Spaces and Affine Spaces
  - Subspaces
  - Linear systems, linear independence, bases, orthonormality
  - Equations for objects in affine spaces
- Cumulative Transformation Matrices (CTM) aka “Composite”, “Current”
  - Translation
  - Rotation
  - Scale
- Parametric Equation of Line Segment
Online Recorded Lectures for CIS 536/636 (Intro to CG)

- **Project Topics for CIS 536/636**
  - Computer Graphics Basics (10)
    - 2. Graphics Pipeline – Week 2
    - 3. Detailed Introduction to Projections and 3-D Viewing – Week 3
    - 4. OpenGL Primer 1 of 3: Basic Primitives and 3-D – Weeks 3-4
    - 5. Rasterizing (Lines, Polygons, Circles, Ellipses) and Clipping – Week 4
    - 6. Lighting and Shading – Week 5
    - 7. OpenGL Primer 2 of 3: Boundaries (Meshes), Transformations – Weeks 5-6
    - 8. Texture Mapping – Week 6
    - 9. OpenGL Primer 3 of 3: Shading and Texturing, VBOs – Weeks 6-7
    - 10. Visible Surface Determination – Week 8

- **Recommended Background Reading for CIS 636**

- **Shared Lectures with CIS 736 (Computer Graphics)**
  - Regular in-class lectures (30) and labs (7)
  - Guidelines for paper reviews – Week 6
  - Preparing term project presentations, CG demos – Weeks 11-12
Background Expected

- Both Courses
  - Proficiency in C/C++ or strong proficiency in Java and ability to learn
  - Strongly recommended: matrix theory or linear algebra (e.g., Math 551)
  - At least 120 hours for semester (up to 150 depending on term project)
  - Angel's *OpenGL: A Primer* recommended

- CIS 536 & 636 *Introduction to Computer Graphics*
  - Fresh background in precalculus: Algebra 1-2, Analytic Geometry
  - Linear algebra basics: matrices, linear bases, vector spaces
  - Watch background lectures

- CIS 736 *Computer Graphics*
  - Recommended: first course in graphics (background lectures as needed)
  - OpenGL experience helps
  - Read up on shaders and shading languages
  - Watch advanced topics lectures; see list before choosing project topic
Math Review for CIS 636

- **Overview: First Month (Weeks 2-5 of Course)**
  - Review of mathematical foundations of CG: analytic geometry, linear algebra
  - Line and polygon rendering
  - Matrix transformations
  - Graphical interfaces

- **Line and Polygon Rendering (Week 3)**
  - Basic line drawing and 2-D clipping
  - Bresenham’s algorithm
  - Follow-up: 3-D clipping, z-buffering *(painter’s algorithm)*

- **Matrix Transformations (Week 4)**
  - Application of linear transformations to rendering
  - Basic operations: translation, rotation, scaling, shearing
  - Follow-up: review of standard graphics libraries (e.g., *OpenGL*)

- **Graphical Interfaces**
  - Brief overview
  - Survey of windowing environments (MFC, Java AWT)
Matrix and Vector Notation

- Vector: Geometric Object with Length (Magnitude), Direction
- Vector Notation (General Form)
  - Row vector
  - Column vector

- Coordinates in $\mathbb{R}^3$ (Euclidean Space)
  - Cartesian (see http://bit.ly/f5z1UC) $\mathbf{a} = (a_x, a_y, a_z)$.
  - Cylindrical (see http://bit.ly/gt5v3u) $\mathbf{v} = (r, \theta, h)$
  - Spherical (see http://bit.ly/f4CvMZ) $\mathbf{v} = (\rho, \theta, \phi)$

- Matrix: Rectangular Array of Numbers

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & 3 \\ 3 & -1 & 2 & -1 \end{bmatrix}$$


Determinants

- **What Are Determinants?**
  - Scalars associated with any square \((k \times k)\) matrix \(M, k \geq 1\)
  - Fundamental meaning: scale coefficient where \(M\) is linear transformation

- **Definitions**
  - \(2 \times 2\) matrix
    \[
    A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
    \]
  - \(2 \times 2\) determinant
    \[
    \det A = ad - bc.
    \]
  - \(3 \times 3\) matrix
    \[
    A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
    \]
  - \(3 \times 3\) determinant
    \[
    \det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)
    \]

  
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- **General case (recursive definition):** see
Vector Operations: Dot & Cross Product, Arithmetic

- **Dot Product aka Inner Product aka Scalar Product**
  \[
  \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
  \]

- **Cross Product**
  \[
  \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
  \]
  \[
  \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}
  \]
  \[
  \mathbf{a} \times \mathbf{b} = ia_2b_3 + ja_3b_1 + ka_1b_2 - ia_3b_2 - ja_1b_3 - ka_2b_1.
  \]

- **Vector Arithmetic**
  \[
  \mathbf{c} = \mathbf{u} + \mathbf{v}
  \]

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Matrix Operations [1]:
Scalar Multiplication & Transpose

- Scalar-Matrix Multiplication

\[ 2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot (-3) \\ 2 \cdot 4 & 2 \cdot (-2) & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix} \]

- Transpose

\[ \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ -6 \\ 3 \\ 7 \end{bmatrix} \]
Matrix Operations [2]:
Addition & Multiplication

Matrix Addition

\[
\begin{bmatrix}
1 & 3 & 1 \\
1 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 5 \\
7 & 5 & 0
\end{bmatrix} = \begin{bmatrix}
1 + 0 & 3 + 0 & 1 + 5 \\
1 + 7 & 0 + 5 & 0 + 0
\end{bmatrix} = \begin{bmatrix}
1 & 3 & 6 \\
8 & 5 & 0
\end{bmatrix}
\]

Matrix Multiplication

\[
A = \begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\
b_{2,1} & b_{2,2} & \cdots & b_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m,1} & b_{m,2} & \cdots & b_{m,p}
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_{1,i} & b_{2,i} & \cdots & b_{m,i}
\end{bmatrix}^T.
\]

\[
AB = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_m
\end{bmatrix}
\begin{bmatrix}
B_1 & B_2 & \cdots & B_p
\end{bmatrix} = \begin{bmatrix}
(A_1 \cdot B_1) & (A_1 \cdot B_2) & \cdots & (A_1 \cdot B_p) \\
(A_2 \cdot B_1) & (A_2 \cdot B_2) & \cdots & (A_2 \cdot B_p) \\
\vdots & \vdots & \ddots & \vdots \\
(A_m \cdot B_1) & (A_m \cdot B_2) & \cdots & (A_m \cdot B_p)
\end{bmatrix}.
\]
Definition: Linear System of Equations (LSE)

- Each of form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.
- System shares same set of variables $x_j$

$$
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
$$

Example

- 3 equations in 3 unknowns

$$
\begin{align*}
    3x + 2y - z &= 1 \\
    2x - 2y + 4z &= -2 \\
    -x + \frac{1}{2}y - z &= 0
\end{align*}
$$

Solution

$$
\begin{align*}
    x &= 1 \\
    y &= -2 \\
    z &= -2
\end{align*}
$$
Cumulative Transformation Matrices: Basic T, R, S

- **T**: Translation (see [http://en.wikipedia.org/wiki/Translation_matrix](http://en.wikipedia.org/wiki/Translation_matrix))
  - **Given**
    - Point to be moved – e.g., vertex of polygon or polyhedron
    - Displacement vector (also represented as point)
  - **Return**: new, displaced (translated) point of rigid body

  - **Given**
    - Point to be rotated about axis
    - Axis of rotation
    - Degrees to be rotated
  - **Return**: new, displaced (rotated) point of rigid body

- **S**: Scaling (see [http://en.wikipedia.org/wiki/Scaling_matrix](http://en.wikipedia.org/wiki/Scaling_matrix))
  - **Given**
    - Set of points centered at origin
    - Scaling factor
  - **Return**: new, displaced (scaled) point

Translation

- **Rigid Body Transformation**
- **To Move \( p \) Distance and Magnitude of Vector \( v \):**

\[
T_v p = \begin{bmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + v_x \\ p_y + v_y \\ p_z + v_z \\ 1 \end{bmatrix} = p + v.
\]

- **Invertibility**

\[T_v^{-1} = T_{-v}.
\]

- **Compositionality**

\[T_u T_v = T_{u+v}.
\]
CG Basics 1 of 10: Math

- Rigid Body Transformation
- Properties: Inverse $\equiv$ Transpose

$$Q^T Q = I = QQ^T$$
$$\det Q = +1$$

- Idea: Define New (Relative) Coordinate System
- Example

$$Q = \begin{bmatrix}
0.6 & -0.8 & 0 \\
0.8 & 0.6 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}$$

- Rotations about x, y, and z Axes (using Plain 3-D Coordinates)

$$Q_x(\theta) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta \\
\end{bmatrix}, \quad Q_y(\theta) = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta \\
\end{bmatrix}, \quad Q_z(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix},$$

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Scaling

- Not Rigid Body Transformation
- Idea: Move Points Toward/Away from Origin

\[
S_{vp} = \begin{bmatrix}
  v_x & 0 & 0 & 0 \\
  0 & v_y & 0 & 0 \\
  0 & 0 & v_z & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
  1
\end{bmatrix} = \begin{bmatrix}
  v_x p_x \\
  v_y p_y \\
  v_z p_z \\
  1
\end{bmatrix}
\]

- Homogeneous Coordinates Make It Easier

\[
S_{vp} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
  1
\end{bmatrix}
\]

- Result

\[
\begin{bmatrix}
  sp_x \\
  sp_y \\
  sp_z \\
  1
\end{bmatrix}
\]

- Ratio Need Not Be Uniform in \( x, y, z \)

Results of \texttt{glScalef(2.0, -0.5, 1.0)}
© 1993 Neider, Davis, Woo
http://fly.cc.fer.hr/~unreal/theredbook/
Other Transformations

- **Shear**: Used with Oblique Projections
- **Perspective to Parallel View Volume** ("D" in Foley et al.)
- **See also**
Quick Review: 
Basic Linear Algebra for CG

- A.1 Vector Spaces and Affine Spaces
  - Equations of lines, planes
  - Vector subspaces and affine subspaces
- A.2 Standard Constructions in Vector Spaces
  - Linear independence and spans
  - Coordinate systems and bases
- A.3 Dot Products and Distances
  - Dot product in $\mathbb{R}^n$
  - Norms in $\mathbb{R}^n$
- A.4 Matrices
  - Binary matrix operations: basic arithmetic
  - Unary matrix operations: transpose and inverse
- Application: Transformations and Change of Coordinate Systems

Affine transformations
© 2005 Trevor McCauley (Senocular)
Vector Spaces and Affine Spaces

**Vector Space**: Set of Points with Addition, Multiplication by Constant

- **Components**
  - Set \( V \) (of vectors \( u, v, w \)) over which addition, scalar multiplication defined
  - Vector addition: \( v + w \)
  - Scalar multiplication: \( \alpha v \)

- **Properties (necessary and sufficient conditions)**
  - Addition: associative, commutative, identity (0 vector such that \( \forall v . 0 + v = v \), admits inverses (\( \forall v . \exists w . v + w = 0 \))
  - Scalar multiplication: satisfies \( \forall \alpha, \beta, v . (\alpha \beta)v = \alpha(\beta v) \), \( \forall v . 1v = v \), \( \forall \alpha, \beta, v . (\alpha + \beta)v = \alpha v + \beta v \), \( \forall \alpha, \beta, v . \alpha(v + w) = \alpha v + \alpha w \)

- **Linear combination**: \( \alpha_1v_1 + \alpha_2v_2 + \ldots + \alpha_nv_n \)

**Affine Space**: Set of Points with Geometric Operations (No “Origin”)

- **Components**
  - Set \( V \) (of points \( P, Q, R \)) and associated vector space
  - Operators: vector difference, point-vector addition

- **Affine combination** (of \( P \) and \( Q \) by \( t \in \mathbb{R} \)): \( P + t(Q - P) \)

- **NB**: for any vector space \( (V, +, \cdot) \) there exists affine space (points \( V \), \( V \))
Linear and Planar Equations in Affine Spaces

- **Equation of Line in Affine Space**
  - Let $P$, $Q$ be points in affine space
  - **Parametric form** (real-valued parameter $t$)
    - Set of points of form $(1 - t)P + tQ$
    - Forms line passing through $P$ and $Q$
  - **Example**
    - Cartesian plane of points $(x, y)$ is an affine space
    - Parametric line between $(a, b)$ and $(c, d)$:
      $$L = \{(1 - t)a + tc, (1 - t)b + td) | t \in \mathbb{R}\}$$

- **Equation of Plane in Affine Space**
  - Let $P$, $Q$, $R$ be points in affine space
  - **Parametric form** (real-valued parameters $s$, $t$)
    - Set of points of form $(1 - s)((1 - t)P + tQ) + sR$
    - Forms plane containing $P$, $Q$, $R$
Vector Space Spans and Affine Spans

**Vector Space Span**
- Definition – set of all linear combinations of a set of vectors
- Example: vectors in \( \mathbb{R}^3 \)
  - Span of single (nonzero) vector \( \mathbf{v} \): line through the origin containing \( \mathbf{v} \)
  - Span of pair of (nonzero, noncollinear) vectors: plane through the origin containing both
  - Span of 3 of vectors in **general position**: all of \( \mathbb{R}^3 \)

**Affine Span**
- Definition – set of all affine combinations of a set of points \( P_1, P_2, ..., P_n \) in an affine space
- Example: vectors, points in \( \mathbb{R}^3 \)
  - Standard affine plan of points \( (x, y, 1)^T \)
  - Consider points \( P, Q \)
  - Affine span: line containing \( P, Q \)
  - Also intersection of span, affine space
Independence

- **Linear Independence**
  - Definition: (linearly) dependent vectors
    - Set of vectors \( \{v_1, v_2, \ldots, v_n\} \) such that one lies in the span of the rest
    - \( \exists v_i \in \{v_1, v_2, \ldots, v_n\} \cdot v_i \in \text{Span} \left( \{v_1, v_2, \ldots, v_n\} \sim \{v_i\} \right) \)
  - (Linearly) independent: \( \{v_1, v_2, \ldots, v_n\} \) not dependent

- **Affine Independence**
  - Definition: (affinely) dependent points
    - Set of points \( \{v_1, v_2, \ldots, v_n\} \) such that one lies in the (affine) span of the rest
    - \( \exists P_i \in \{P_1, P_2, \ldots, P_n\} \cdot P_i \in \text{Span} \left( \{P_1, P_2, \ldots, P_n\} \sim \{P_i\} \right) \)
  - (Affinely) independent: \( \{P_1, P_2, \ldots, P_n\} \) not dependent

- **Consequences of Linear Independence**
  - Equivalent condition: \( \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \iff \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \)
  - Dimension of span is equal to the number of vectors
Subspaces

- Intuitive Idea
  - $\mathbb{R}^n$: vector or affine space of “equal or lower dimension”
  - Closed under constructive operator for space

- Linear Subspace
  - Definition
    - Subset $S$ of vector space $(V, +, \cdot)$
    - Closed under addition (+) and scalar multiplication (·)
  - Examples
    - Subspaces of $\mathbb{R}^3$: origin $(0, 0, 0)$, line through the origin, plane containing origin, $\mathbb{R}^3$ itself
    - For vector $v$, $\{\alpha v \mid \alpha \in \mathbb{R}\}$ is a subspace (why?)

- Affine Subspace
  - Definition
    - Nonempty subset $S$ of vector space $(V, +, \cdot)$
    - Closure $S'$ of $S$ under point subtraction is a linear subspace of $V$
  - Important affine subspace of $\mathbb{R}^4$: $\{(x, y, z, 1)\}$
  - Foundation of homogeneous coordinates, 3-D transformations
Spanning Set (of Set $S$ of Vectors)
- Definition: set of vectors for which any vector in $\text{Span}(S)$ can be expressed as linear combination of vectors in spanning set
- Intuitive idea: spanning set “covers” $\text{Span}(S)$

Basis (of Set $S$ of Vectors)
- Definition
  - Minimal spanning set of $S$
  - Minimal: any smaller set of vectors has smaller span
- Alternative definition: linearly independent spanning set

Exercise
- Claim: basis of subspace of vector space is always linearly independent
- Proof: by contradiction (suppose basis is dependent... not minimal)

Standard Basis for $\mathbb{R}^3$: $i$, $j$, $k$
- $E = \{e_1, e_2, e_3\}$, $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, $e_3 = (0, 0, 1)^T$
- How to use this as coordinate system?
Coordinates and Coordinate Systems

- Coordinates Using Bases
  - Coordinates
    - Consider basis \( B = \{v_1, v_2, \ldots, v_n\} \) for vector space
    - Any vector \( v \) in the vector space can be expressed as linear combination of vectors in \( B \)
    - **Definition**: coefficients of linear combination are coordinates
  - Example
    - \( E = \{e_1, e_2, e_3\} \), \( i = e_1 = (1, 0, 0)^T \), \( j = e_2 = (0, 1, 0)^T \), \( k = e_3 = (0, 0, 1)^T \)
    - Coordinates of \((a, b, c)\) with respect to \( E \): \((a, b, c)^T\)
- Coordinate System
  - **Definition**: set of independent points in affine space
  - **Affine span of coordinate system is entire affine space**
- Exercise
  - Derive basis for associated vector space of arbitrary coordinate system
  - (Hint: consider definition of affine span...)
Dot Products and Distances

- **Dot Product in \( \mathbb{R}^n \)**
  - **Given:** vectors \( u = (u_1, u_2, \ldots, u_n)^T \), \( v = (v_1, v_2, \ldots, v_n)^T \)
  - **Definition**
    - Dot product \( u \cdot v \equiv u_1v_1 + u_2v_2 + \cdots + u_nv_n \)
    - Also known as inner product
    - In \( \mathbb{R}^n \), called scalar product

- **Applications of the Dot Product**
  - Normalization of vectors
  - Distances
  - Generating equations
  - See Appendix A.3, Foley et al. (FVFH aka FVD)
Norms and Distance Formulas

- **Length**
  - **Definition**
    \[ \| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \]
    \[ \mathbf{v} \cdot \mathbf{v} = \sum_i v_i^2 \]
  - *aka Euclidean norm*

- **Applications of the Dot Product**
  - **Normalization of vectors:** division by scalar length \( \| \mathbf{v} \| \) converts to **unit vector**
  - **Distances**
    \[ \| \mathbf{Q} - \mathbf{P} \| \]
    \[ \text{From points to planes} \]
  - **Generating equations (e.g., point loci):** circles, hollow cylinders, etc.
  - **Ray / object intersection equations**
  - *See A.3.5, FVD*
Orthonormal Bases

- **Orthogonality**
  - Given: vectors \( u = (u_1, u_2, \ldots, u_n)^T, v = (v_1, v_2, \ldots, v_n)^T \)
  - **Definition**
    - \( u, v \) are orthogonal if \( u \cdot v = 0 \)
    - In \( \mathbb{R}^2 \), angle between orthogonal vectors is 90°

- **Orthonormal Bases**
  - **Necessary and sufficient conditions**
    - \( B = \{b_1, b_2, \ldots, b_n\} \) is basis for given vector space
    - Every pair \( (b_i, b_j) \) is orthogonal
    - Every vector \( b_i \) is of unit magnitude (\( ||v_i|| = 1 \))
  - **Convenient property**: can just take dot product \( v \cdot b_i \) to find coefficients in linear combination (coordinates with respect to \( B \)) for vector \( v \)
Parametric Equation of a Line Segment

- Parametric form for line segment
  
  \[ X = x_0 + t(x_1 - x_0) \quad 0 \leq t \leq 1 \]
  
  \[ Y = y_0 + t(y_1 - y_0) \]
  
  \[ P(t) = P_0 + t(P_1 - P_0) \]

- “true,” i.e., interior intersection, if \( s \)edge and \( t \)line in \([0,1]\)
Rotation as Change of Basis

- 3 x 3 rotation matrices
- We learned about 3 x 3 matrices that “rotate” the world (we’re leaving out the homogeneous coordinate for simplicity)
- When they do, the three unit vectors that used to point along the $x$, $y$, and $z$ axes are moved to new positions
- Because it is a rigid-body rotation
  * the new vectors are still unit vectors
  * the new vectors are still perpendicular to each other
  * the new vectors still satisfy the “right hand rule”
- Any matrix transformation that has these three properties is a rotation about some axis by some amount!
- Let’s call three $x$-axis, $y$-axis, and $z$-axis-aligned unit vectors $e_1$, $e_2$, $e_3$
- Writing out:

$$
e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
Textbook and Recommended Books

Required Textbook


Recommended References


Summary

- **Cumulative Transformation Matrices (CTM): T, R, S**
  - Translation
  - Rotation
  - Scaling
  - Setup for Shear, Perspective to Parallel – see Eberly, Foley et al.
- “Matrix Stack” in OpenGL: Premultiplication of Matrices
- **Coming Up**
  - Parametric equations in clipping
  - Intersection testing: ray-cube, ray-sphere, implicit equations (ray tracing)
- **Homogeneous Coordinates: What Is That 4th Coordinate?**
  - Crucial for ease of normalizing T, R, S transformations in graphics
  - See: Slide 22 of this lecture
  - Note: Slides 13 & 15 (T, S) versus 14 (R)
  - Read about them in Eberly 2e, Angel 3e
  - Special case: barycentric coordinates
Terminology

- **Cumulative Transformation Matrices (CTM):** Translation, Rotation, Scaling
- **Some Basic Analytic Geometry and Linear Algebra for CG**
  - **Vector space** (VS) – set of vectors admitting addition, scalar multiplication and observing VS axioms
  - **Affine space** (AS) – set of points with associated vector space admitting vector difference, point-vector addition and observing AS axioms
  - **Linear subspace** – nonempty subset $S$ of VS $(V, +, \cdot)$ closed under $+$ and $\cdot$
  - **Affine subspace** – nonempty subset $S$ of VS $(V, +, \cdot)$ such that closure $S'$ of $S$ under point subtraction is a linear subspace of $V$
  - **Span** – set of all linear combinations of set of vectors
  - **Linear independence** – property of set of vectors that none lies in span of others
  - **Basis** – minimal spanning set of set of vectors
  - **Dot product** – scalar-valued inner product $\langle u, v \rangle \equiv u \cdot v \equiv u_1v_1 + u_2v_2 + \ldots + u_nv_n$
  - **Orthogonality** – property of vectors $u$, $v$ that $u \cdot v = 0$
  - **Orthonormality** – basis containing pairwise-orthogonal unit vectors
  - **Length (Euclidean norm)** – $\|v\| = \sqrt{v \cdot v}$