$h(n) = W(n)$ and unit arc costs. (Note that $W(n)$ is a lower bound on the number of steps remaining to the goal.) It is reasonable to say that **A*** with $h(n) = W(n)$ is more informed than breadth-first search, which uses $h(n) \equiv 0$.

We would expect intuitively that the more informed algorithm typically would need to expand fewer nodes to find a minimal cost path. In the case of the 8-puzzle, this observation is supported by comparing Figure 2.7 with Figure 2.8. Of course, merely because one algorithm expands fewer nodes than another does not imply that it is more efficient. The more informed algorithm may indeed have to make more costly computations, which would destroy efficiency. Nevertheless, the number of nodes expanded by an algorithm is one of the factors that determines efficiency, and it is a factor that permits simple comparisons.

Suppose that $\mathbf{A}_2$ is more informed than $\mathbf{A}_1$ and that both $\mathbf{A}_1$ and $\mathbf{A}_2$ are versions of **A***. Suppose that $\mathbf{A}_1$ and $\mathbf{A}_2$ are used to search an implicit graph having a path from a given node s to a goal node. Both, of course, will terminate in an optimal path. We will show that, at termination, if node n in G was expanded by $\mathbf{A}_2$, it was also expanded by $\mathbf{A}_1$. Thus, $\mathbf{A}_1$ always expands at least as many nodes as does the more informed $\mathbf{A}_2$.

We prove this result using induction on the depth of a node in the $\mathbf{A}_2$ search tree at termination. First, we prove that if $\mathbf{A}_2$ expands a node n having zero depth in its search tree, then so will $\mathbf{A}_1$. But, in this case, $n = s$. If s is a goal node, neither algorithm expands any nodes. If s is not a goal node, both algorithms expand node s. Continuing the inductive argument, we assume (the induction hypothesis) that $\mathbf{A}_1$ expands all the nodes expanded by $\mathbf{A}_2$ having depth k, or less, in the $\mathbf{A}_2$ search tree. We must now prove that any node n expanded by $\mathbf{A}_2$ and of depth $k + 1$ in the $\mathbf{A}_2$ search tree is also expanded by $\mathbf{A}_1$. By the induction hypothesis, any ancestor of n in the $\mathbf{A}_2$ search tree is also expanded by $\mathbf{A}_1$. Thus, node n is in the $\mathbf{A}_1$ search tree and there is a path from s to n in the $\mathbf{A}_1$ search tree that is no more costly than the cost of the path from s to n in the $\mathbf{A}_2$ search tree; that is,

$$g_1(n) \leq g_2(n) .$$

Let us suppose the opposite of what we are trying to prove, namely, that $\mathbf{A}_1$ did not expand node n expanded by $\mathbf{A}_2$. Certainly, at termination of $\mathbf{A}_1$, node n must be on *OPEN* for $\mathbf{A}_1$, because $\mathbf{A}_1$ expanded a parent of node n. Since $\mathbf{A}_1$ terminated in a minimal cost path without expanding node n, we know that

$$f_1(n) \geq f^*(s) ,$$

thus,

$$g_1(n) + h_1(n) \geq f^*(s) .$$

Since we have already shown that $g_1(n) \leq g_2(n)$, we have

$$h_1(n) \geq f^*(s) - g_2(n) .$$

But, by RESULT 5, since $\mathbf{A}_2$ expanded node n, we have

$$f_2(n) \leq f^*(s)$$

or

$$g_2(n) + h_2(n) \leq f^*(s)$$

or

$$h_2(n) \leq f^*(s) - g_2(n) .$$

Comparing this inequality for $h_2(n)$ with the earlier one for $h_1(n)$ (i.e., $h_1(n) \geq f^*(s) - g_2(n)$) reveals that, at least at node n, h_1 must be as large as h_2, which violates the assumption that $\mathbf{A}_2$ is more informed than $\mathbf{A}_1$. Thus, we have

RESULT 6: If $\mathbf{A}_1$ and $\mathbf{A}_2$ are two versions of **A*** such that $\mathbf{A}_2$ is more informed than $\mathbf{A}_1$, then at the termination of their searches on any graph having a path from s to a goal node, every node expanded by $\mathbf{A}_2$ is also expanded by $\mathbf{A}_1$. It follows that $\mathbf{A}_1$ expands at least as many nodes as does $\mathbf{A}_2$.

2.4.5. THE MONOTONE RESTRICTION

Describing the **GRAPHSEARCH** procedure, we noted that when a node n is expanded, some of its successors may already be on *OPEN* or *CLOSED*. The search tree may then need to be adjusted so that it defines