CG Basics 1 of 10

Mathematical Foundations: Vectors, Matrices, & Parametric Equations

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KSOL course page: http://bit.ly/hGvXlH
Course web site: http://www.kddresearch.org/Courses/CIS636
Instructor home page: http://www.cis.ksu.edu/~bhsu

Readings:
Sections 2.1 – 2.2, 13.2, 14.1 – 14.4, 17.1, Eberly 2e – see http://bit.ly/ieUq45
Appendices 1-4, Foley, J. D., VanDam, A., Feiner, S. K., & Hughes, J. F. (1991). *Computer Graphics, Principles and Practice, Second Edition in C.*
McCauley (Senocular.com) tutorial: http://bit.ly/2yNPD

Lecture Outline

- Quick Review: Basic Precalculus and Linear Algebra for CG
- Matrix and Vector Notation, Operations
- Precalculus: Analytic Geometry and Trigonometry
  - Dot products and distance measures (norms, equations)
  - Review of some basic trigonometry concepts
- Vector Spaces and Affine Spaces
  - Subspaces
  - Linear systems, linear independence, bases, orthonormality
  - Equations for objects in affine spaces
- Cumulative Transformation Matrices (CTM) *aka* "Composite", "Current"
  - Translation
  - Rotation
  - Scale
- Parametric Equation of Line Segment

Online Recorded Lectures for CIS 536/636 (Intro to CG)

- Project Topics for CIS 536/636
- Computer Graphics Basics (10)
  - 1. Mathematical Foundations – Week 1 - 2
  - 2. Graphics Pipeline – Week 2
  - 3. Detailed Introduction to Projections and 3-D Viewing – Week 3
  - 4. OpenGL Primer 1 of 3: Basic Primitives and 3-D – Weeks 3-4
  - 5. Rasterizing (Lines, Polygons, Circles, Ellipses) and Clipping – Week 4
  - 6. Lighting and Shading – Week 5
  - 7. OpenGL Primer 2 of 3: Boundaries (Meshes), Transformations – Weeks 5-6
  - 8. Texture Mapping – Week 6
  - 9. OpenGL Primer 3 of 3: Shading and Texturing, VBOs – Weeks 6-7
  - 10. Visible Surface Determination – Week 8
- Recommended Background Reading for CIS 636
- Shared Lectures with CIS 736 (*Computer Graphics*)
  - Regular in-class lectures (30) and labs (7)
  - Guidelines for paper reviews – Week 6
  - Preparing term project presentations, CG demos – Weeks 11-12

Background Expected

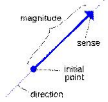
- Both Courses
  - Proficiency in C/C++ or *strong* proficiency in Java and ability to learn
  - Strongly recommended: matrix theory or linear algebra (e.g., Math 551)
  - At least 120 hours for semester (up to 150 depending on term project)
  - Textbook: *3D Game Engine Design*, *Second Edition* (2006), Eberly
  - Angel's *OpenGL: A Primer* recommended
- CIS 536 & 636 *Introduction to Computer Graphics*
  - Fresh background in precalculus: Algebra 1-2, Analytic Geometry
  - Linear algebra basics: matrices, linear bases, vector spaces
  - Watch background lectures
- CIS 736 *Computer Graphics*
  - Recommended: first course in graphics (background lectures as needed)
  - OpenGL experience helps
  - Read up on shaders and shading languages
  - Watch advanced topics lectures; see list before choosing project topic

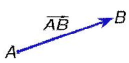
Math Review for CIS 636

- Overview: First Month (Weeks 2-5 of Course)
  - Review of mathematical foundations of CG: analytic geometry, linear algebra
  - Line and polygon rendering
  - Matrix transformations
  - Graphical interfaces
- Line and Polygon Rendering (Week 3)
  - Basic line drawing and 2-D clipping
  - Bresenham's algorithm
  - Follow-up: 3-D clipping, z-buffering (painter's algorithm)
- Matrix Transformations (Week 4)
  - Application of linear transformations to rendering
  - Basic operations: translation, rotation, scaling, shearing
  - Follow-up: review of standard graphics libraries (e.g., *OpenGL*)
- Graphical Interfaces
  - Brief overview
  - Survey of windowing environments (MFC, Java AWT)

Matrix and Vector Notation

- Vector: Geometric Object with Length (Magnitude), Direction
- Vector Notation (General Form)
  - Row vector $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$
  - Column vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$

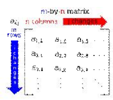




Wikipedia: *Vector*
http://bit.ly/eBrl09

- Coordinates in $\mathbb{R}^3$ (Euclidean Space)
  - Cartesian (see http://bit.ly/f5z1UC) $\mathbf{a} = (a_x, a_y, a_z).$
  - Cylindrical (see http://bit.ly/gt5v3u) $\mathbf{v} = (r, \angle\theta, h)$
  - Spherical (see http://bit.ly/f4CvMZ) $\mathbf{v} = (\rho, \angle\theta, \angle\phi)$
- Matrix: Rectangular Array of Numbers

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$



Wikipedia: *Matrix (mathematics)*
http://bit.ly/fwpDwd

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Determinants

- **What Are Determinants?**
  - Scalars associated with any square ($k \times k$) matrix *M*, $k \geq 1$
  - Fundamental meaning: scale coefficient where *M* is linear transformation
- **Definitions**
  - 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - 2 × 2 determinant $\det A = ad - bc.$
  - 3 × 3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
  - 3 × 3 determinant

$$\det A = aei + bfg + cdh - afh - bdi - ceg.$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

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  - General case (recursive definition): see http://mathworld.wolfram.com/Determinant.html

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Vector Operations: Dot & Cross Product, Arithmetic

- **Dot Product *aka* Inner Product *aka* Scalar Product**

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

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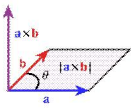
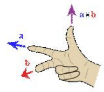


$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}a_2b_3 + \mathbf{j}a_3b_1 + \mathbf{k}a_1b_2 - \mathbf{i}a_3b_2 - \mathbf{j}a_1b_3 - \mathbf{k}a_2b_1.$$



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$$c\mathbf{v}$$
$$\mathbf{u} + \mathbf{v}$$

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Matrix Operations [1]: Scalar Multiplication & Transpose

- **Scalar-Matrix Multiplication**

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 8 & 2\cdot -3 \\ 2\cdot 4 & 2\cdot -2 & 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

- **Transpose**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

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Matrix Operations [2]: Addition & Multiplication

- **Matrix Addition**

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

- **Matrix Multiplication**

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,p} \end{bmatrix} = [B_1 \; B_2 \; \cdots \; B_p]$$

$$A_i = [a_{i,1} \; a_{i,2} \; \cdots \; a_{i,n}] \qquad B_i = [b_{1,i} \; b_{2,i} \; \cdots \; b_{n,i}]^T.$$

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} [B_1 \; B_2 \; \ldots \; B_p] = \begin{bmatrix} (A_1 \cdot B_1) & (A_1 \cdot B_2) & \ldots & (A_1 \cdot B_p) \\ (A_2 \cdot B_1) & (A_2 \cdot B_2) & \ldots & (A_2 \cdot B_p) \\ \vdots & \vdots & \ddots & \vdots \\ (A_m \cdot B_1) & (A_m \cdot B_2) & \ldots & (A_m \cdot B_p) \end{bmatrix}.$$

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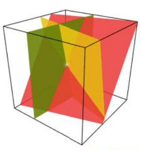
Linear Systems of Equations

- **Definition: Linear System of Equations (LSE)**
  - Collection of linear equations (see http://bit.ly/dNa2MO)
  - Each of form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$
  - System shares same set of variables x_i

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

- **Example**
  - 3 equations in 3 unknov

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - z &= 0 \end{aligned}$$



  - Solution

$$x = 1 \quad y = -2 \quad z = -2$$

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Cumulative Transformation Matrices: Basic T, R, S

- **T: Translation (see http://en.wikipedia.org/wiki/Translation_matrix)**
  - Given
    - ⇨ Point to be moved – e.g., vertex of polygon or polyhedron
    - ⇨ Displacement vector (also represented as point)
  - Return: new, displaced (translated) point of rigid body
- **R: Rotation (see http://en.wikipedia.org/wiki/Rotation_matrix)**
  - Given
    - ⇨ Point to be rotated about axis
    - ⇨ Axis of rotation
    - ⇨ Degrees to be rotated
  - Return: new, displaced (rotated) point of rigid body
- **S: Scaling (see http://en.wikipedia.org/wiki/Scaling_matrix)**
  - Given
    - ⇨ Set of points centered at origin
    - ⇨ Scaling factor
  - Return: new, displaced (scaled) point
- **General: http://en.wikipedia.org/wiki/Transformation_matrix**

Translation

- Rigid Body Transformation
- To Move p Distance and Magnitude of Vector v:

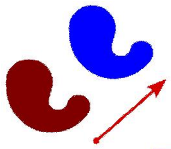
$$T_{\mathbf{v}}\mathbf{p} = \begin{bmatrix} 1 & 0 & 0 & v_x \\ 0 & 1 & 0 & v_y \\ 0 & 0 & 1 & v_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + v_x \\ p_y + v_y \\ p_z + v_z \\ 1 \end{bmatrix} = \mathbf{p} + \mathbf{v}.$$

- Invertibility

$$T_{\mathbf{v}}^{-1} = T_{-\mathbf{v}}.$$

- Compositionality

$$T_{\mathbf{u}} T_{\mathbf{v}} = T_{\mathbf{u}+\mathbf{v}}.$$



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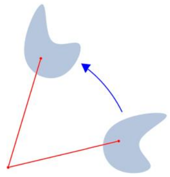
Rotation

- Rigid Body Transformation
- Properties: Inverse ≡ Transpose

$$Q^T Q = I = QQ^T$$
$$\det Q = +1$$



- Idea: Define New (Relative) Coordinate System
- Example

$$Q = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotations about x, y, and z Axes (using Plain 3-D Coordinates)

$$Q_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad Q_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad Q_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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Scaling

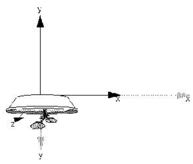
- Not Rigid Body Transformation
- Idea: Move Points Toward/Away from Origin

$$S_v p = \begin{bmatrix} v_x & 0 & 0 & 0 \\ 0 & v_y & 0 & 0 \\ 0 & 0 & v_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} v_x p_x \\ v_y p_y \\ v_z p_z \\ 1 \end{bmatrix}$$

Results of glScalef(2.0, -0.5, 1.0)
© 1993 Neider, Davis, Woo
http://fly.cc.fer.hr/~unreal/theredbook/

- Homogeneous Coordinates Make It Easier

$$S_v p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \frac{1}{s} \end{bmatrix}$$



- Result

$$\begin{bmatrix} sp_x \\ sp_y \\ sp_z \\ 1 \end{bmatrix}$$

- Ratio Need Not Be Uniform in *x*, *y*, *z*

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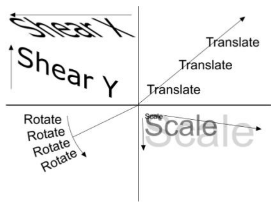
Other Transformations

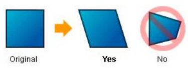
- Shear: Used with Oblique Projections
- Perspective to Parallel View Volume ("*D*" in Foley *et al.*)
- See also
  - http://en.wikipedia.org/wiki/Transformation_matrix
  - http://www.senocular.com/flash/tutorials/transformmatrix/



http://www.bobpowell.net/transformations.htm

Quick Review: Basic Linear Algebra for CG

- Reference: Appendix A.1 – A.4, Foley *et al.*
- A.1 Vector Spaces and Affine Spaces
  - Equations of lines, planes
  - Vector subspaces and affine subspaces
- A.2 Standard Constructions in Vector Spaces
  - Linear independence and spans
  - Coordinate systems and bases
- A.3 Dot Products and Distances
  - Dot product in $\mathbb{R}^n$
  - Norms in $\mathbb{R}^n$
- A.4 Matrices
  - Binary matrix operations: basic arithmetic
  - Unary matrix operations: transpose and inverse
- Application: Transformations and Change of Coordinate Systems



Affine transformations
© 2005 Trevor McCauley
(Senocular)

Vector Spaces and Affine Spaces

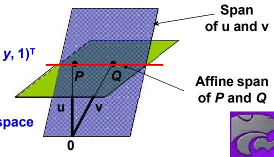
- Vector Space: Set of Points with Addition, Multiplication by Constant
  - Components
    - ⇨ Set *V* (of vectors **u**, **v**, **w**) over which addition, scalar multiplication defined
    - ⇨ Vector addition: **v** + **w**
    - ⇨ Scalar multiplication: α**v**
  - Properties (necessary and sufficient conditions)
    - ⇨ Addition: associative, commutative, identity (**0** vector such that ∀ **v** . **0** + **v** = **v**), admits inverses (∀ **v** . ∃**w** . **v** + **w** = **0**)
    - ⇨ Scalar multiplication: satisfies ∀ α, β, **v** . (αβ)**v** = α(β**v**), ∀**v** . 1**v** = **v**, ∀ α, β, **v** . (α + β)**v** = α**v** + β**v**, ∀ α, β, **v** . α(**v** + **w**) = α**v** + α**w**
  - Linear combination: $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$
- Affine Space: Set of Points with Geometric Operations (No "Origin")
  - Components
    - ⇨ Set *V* (of points *P*, *Q*, *R*) *and* associated vector space
    - ⇨ Operators: vector difference, point-vector addition
  - Affine combination (of *P* and *Q* by $t \in \mathbb{R}$): $P + t(Q - P)$
  - *NB*: for any vector space (*V*, +, ·) there exists affine space (points(*V*), *V*)

Linear and Planar Equations in Affine Spaces

- **Equation of Line in Affine Space**
  - Let P, Q be points in affine space
  - Parametric form (real-valued parameter t)
    - ⇨ Set of points of form $(1 - t)P + tQ$
    - ⇨ Forms line passing through P and Q
  - Example
    - ⇨ Cartesian plane of points (x, y) is an affine space
    - ⇨ Parametric line between (a, b) and (c, d):
      $L = \{((1 - t)a + tc, (1 - t)b + td) \mid t \in \mathbb{R}\}$
- **Equation of Plane in Affine Space**
  - Let P, Q, R be points in affine space
  - Parametric form (real-valued parameters s, t)
    - ⇨ Set of points of form $(1 - s)((1 - t)P + tQ) + sR$
    - ⇨ Forms plane containing P, Q, R

Vector Space Spans and Affine Spans

- **Vector Space Span**
  - Definition – set of all linear combinations of a set of vectors
  - Example: vectors in $\mathbb{R}^3$
    - ⇨ Span of single (nonzero) vector **v**: line through the origin containing **v**
    - ⇨ Span of pair of (nonzero, noncollinear) vectors: plane through the origin containing both
    - ⇨ Span of 3 of vectors in general position: all of $\mathbb{R}^3$
- **Affine Span**
  - Definition – set of all affine combinations of a set of points $P_1, P_2, \ldots, P_n$ in an affine space
  - Example: vectors, points in $\mathbb{R}^3$
    - ⇨ Standard affine plan of points $(x, y, 1)^T$
    - ⇨ Consider points P, Q
    - ⇨ Affine span: line containing P, Q
    - ⇨ Also intersection of span, affine space



Independence

- **Linear Independence**
  - Definition: (linearly) dependent vectors
    - ⇨ Set of vectors $\{v_1, v_2, \ldots, v_n\}$ such that one lies in the span of the rest
    - ⇨ $\exists\, v_i \in \{v_1, v_2, \ldots, v_n\} \,.\, v_i \in \text{Span}(\{v_1, v_2, \ldots, v_n\} \sim \{v_i\})$
  - (Linearly) independent: $\{v_1, v_2, \ldots, v_n\}$ not dependent
- **Affine Independence**
  - Definition: (affinely) dependent points
    - ⇨ Set of points $\{v_1, v_2, \ldots, v_n\}$ such that one lies in the (affine) span of the rest
    - ⇨ $\exists\, P_i \in \{P_1, P_2, \ldots, P_n\} \,.\, P_i \in \text{Span}(\{P_1, P_2, \ldots, P_n\} \sim \{P_i\})$
  - (Affinely) independent: $\{P_1, P_2, \ldots, P_n\}$ not dependent
- **Consequences of Linear Independence**
  - Equivalent condition: $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$
  - Dimension of span is equal to the number of vectors

Subspaces

- **Intuitive Idea**
  - $\mathbb{R}^n$: vector or affine space of "equal or lower dimension"
  - Closed under constructive operator for space
- **Linear Subspace**
  - Definition
    - ⇨ Subset S of vector space $(V, +, \cdot)$
    - ⇨ Closed under addition (+) and scalar multiplication (·)
  - Examples
    - ⇨ Subspaces of $\mathbb{R}^3$: origin (0, 0, 0), line through the origin, plane containing origin, $\mathbb{R}^3$ itself
    - ⇨ For vector **v**, $\{\alpha v \mid \alpha \in \mathbb{R}\}$ is a subspace (why?)
- **Affine Subspace**
  - Definition
    - ⇨ Nonempty subset S of vector space $(V, +, \cdot)$
    - ⇨ Closure S' of S under point subtraction is a linear subspace of V
  - *Important affine subspace of $\mathbb{R}^4$: $\{(x, y, z, 1)\}$*
  - Foundation of homogeneous coordinates, 3-D transformations

Bases

- **Spanning Set (of Set S of Vectors)**
  - Definition: set of vectors for which any vector in Span(S) can be expressed as linear combination of vectors in spanning set
  - Intuitive idea: spanning set "covers" Span(S)
- **Basis (of Set S of Vectors)**
  - Definition
    - ⇨ Minimal spanning set of S
    - ⇨ Minimal: any smaller set of vectors has smaller span
  - Alternative definition: linearly independent spanning set
- **Exercise**
  - Claim: basis of subspace of vector space is always linearly independent
  - Proof: by contradiction (suppose basis is dependent … not minimal)
- **Standard Basis for $\mathbb{R}^3$: i, j, k**
  - $E = \{e_1, e_2, e_3\}$, $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, $e_3 = (0, 0, 1)^T$
  - *How to use this as coordinate system?*

Coordinates and Coordinate Systems

- **Coordinates Using Bases**
  - Coordinates
    - ⇨ Consider basis $B = \{v_1, v_2, \ldots, v_n\}$ for vector space
    - ⇨ Any vector **v** in the vector space can be expressed as linear combination of vectors in **B**
    - ⇨ Definition: coefficients of linear combination are coordinates
  - Example
    - ⇨ $E = \{e_1, e_2, e_3\}$, $i \equiv e_1 = (1, 0, 0)^T$, $j \equiv e_2 = (0, 1, 0)^T$, $k \equiv e_3 = (0, 0, 1)^T$
    - ⇨ Coordinates of (a, b, c) with respect to **E**: $(a, b, c)^T$
- **Coordinate System**
  - Definition: set of independent points in affine space
  - Affine span of coordinate system is entire affine space
- **Exercise**
  - Derive basis for associated vector space of arbitrary coordinate system
  - (Hint: consider definition of affine span …)

Dot Products and Distances

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- **Dot Product in $\mathbb{R}^n$**
  - Given: vectors $u = (u_1, u_2, \ldots, u_n)^T$, $v = (v_1, v_2, \ldots, v_n)^T$
  - Definition
    - Dot product $u \cdot v \equiv u_1v_1 + u_2v_2 + \ldots + u_nv_n$
    - Also known as inner product
    - In $\mathbb{R}^n$, called scalar product
- **Applications of the Dot Product**
  - Normalization of vectors
  - Distances
  - Generating equations
  - See Appendix A.3, Foley *et al.* (FVFH *aka* FVD)

Norms and Distance Formulas

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- **Length**
  - Definition
    - $\|v\| = \sqrt{v \bullet v}$
    - $v \cdot v = \sum_i v_i^2$
  - *aka* Euclidean norm
- **Applications of the Dot Product**
  - Normalization of vectors: division by scalar length $\| v \|$ converts to unit vector
  - Distances
    - Between points: $\| Q - P \|$
    - From points to planes
  - Generating equations (e.g., point loci): circles, hollow cylinders, etc.
  - Ray / object intersection equations
  - See A.3.5, FVD

Orthonormal Bases

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- **Orthogonality**
  - Given: vectors $u = (u_1, u_2, \ldots, u_n)^T$, $v = (v_1, v_2, \ldots, v_n)^T$
  - Definition
    - u, v are orthogonal if $u \cdot v = 0$
    - In $\mathbb{R}^2$, angle between orthogonal vectors is 90°
- **Orthonormal Bases**
  - Necessary and sufficient conditions
    - $B = \{b_1, b_2, \ldots, b_n\}$ is basis for given vector space
    - Every pair (b_i, b_j) is orthogonal
    - Every vector b_i is of unit magnitude ($\| v_i \| = 1$)
  - Convenient property: can just take dot product $v \cdot b_i$ to find coefficients in linear combination (coordinates with respect to B) for vector v

Parametric Equation of a Line Segment

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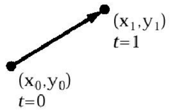
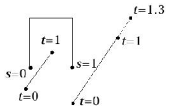
- Parametric form for line segment
  - $X = x_0 + t(x_1 - x_0)$ $0 \le t \le 1$
  - $Y = y_0 + t(y_1 - y_0)$
  - $P(t) = P_0 + t(P_1 - P_0)$



- "true," i.e., interior intersection, if *sedge* and *tline* in [0,1]



Rotation as Change of Basis

29

- 3 x 3 rotation matrices
- We learned about 3 x 3 matrices that "rotate" the world (we're leaving out the homogeneous coordinate for simplicity)
- When they do, the three unit vectors that used to point along the *x*, *y*, and *z* axes are moved to new positions
- Because it is a rigid-body rotation
  - the new vectors are still unit vectors
  - the new vectors are still perpendicular to each other
  - the new vectors still satisfy the "right hand rule"
- *Any* matrix transformation that has these three properties is a rotation about *some* axis by *some* amount!
- Let's call three *x*-axis, *y*-axis, and *z*-axis-aligned unit vectors e_1, e_2, e_3
- Writing out:

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Textbook and Recommended Books

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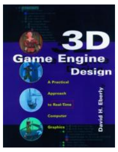
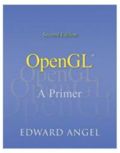
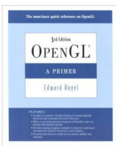


1st edition (outdated)

2nd edition



2nd edition (OK to use)



3rd edition

Required Textbook

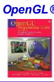
Eberly, D. H. (2006). *3D Game Engine Design: A Practical Approach to Real-Time Computer Graphics, second edition.* San Francisco, CA: Morgan Kauffman.

Recommended References

Angel, E. O. (2007). *OpenGL: A Primer, third edition*. Reading, MA: Addison-Wesley. [2nd edition on reserve]

Shreiner, D., Woo, M., Neider, J., & Davis, T. (2007). *OpenGL® Programming Guide: The Official Guide to Learning OpenGL®, Version 2.1, sixth edition*.

["The Red Book": use 5th ed. or later]



Summary

- Cumulative Transformation Matrices (CTM): T, R, S
  - Translation
  - Rotation
  - Scaling
  - Setup for Shear, Perspective to Parallel – see Eberly, Foley *et al.*
- "Matrix Stack" in OpenGL: Premultiplication of Matrices
- Coming Up
  - Parametric equations in clipping
  - Intersection testing: ray-cube, ray-sphere, implicit equations (ray tracing)
- Homogeneous Coordinates: What Is That 4th Coordinate?
  - http://en.wikipedia.org/wiki/Homogeneous_coordinates
  - Crucial for ease of normalizing T, R, S transformations in graphics
  - See: Slide 22 of this lecture
  - Note: Slides 13 & 15 (T, S) versus 14 (R)
  - Read about them in Eberly *2e*, Angel *3e*
  - Special case: barycentric coordinates

Terminology

- Cumulative Transformation Matrices (CTM): Translation, Rotation, Scaling
- Some Basic Analytic Geometry and Linear Algebra for CG
  - Vector space (VS) – set of vectors admitting addition, scalar multiplication and observing VS axioms
  - Affine space (AS) – set of points with associated vector space admitting vector difference, point-vector addition and observing AS axioms
  - Linear subspace – nonempty subset *S* of VS (*V*, +, ·) closed under + and ·
  - Affine subspace – nonempty subset *S* of VS (*V*, +, ·) such that closure *S*' of *S* under point subtraction is a linear subspace of *V*
  - Span – set of all linear combinations of set of vectors
  - Linear independence – property of set of vectors that none lies in span of others
  - Basis – minimal spanning set of set of vectors
  - Dot product – scalar-valued inner product $\langle u, v \rangle \equiv u \bullet v \equiv u_1v_1 + u_2v_2 + \ldots + u_nv_n$
  - Orthogonality – property of vectors u, v that $u \bullet v = 0$
  - Orthonormality – basis containing pairwise-orthogonal unit vectors
  - Length (Euclidean norm) – $\|v\| = \sqrt{v \bullet v}$